A Nested Error Regression Model with High Dimensional Parameter for Small Area Estimation

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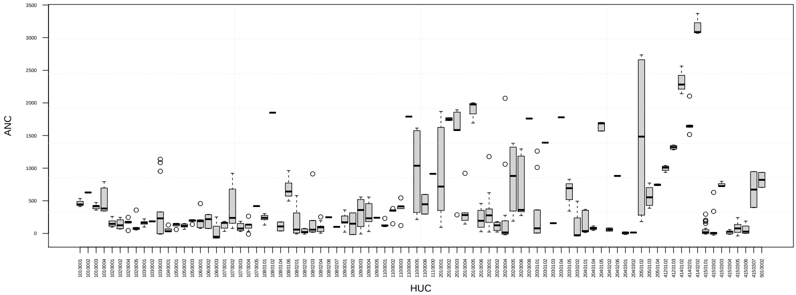
Based on my joint with Nicola Salvati, University of Pisa, Italy

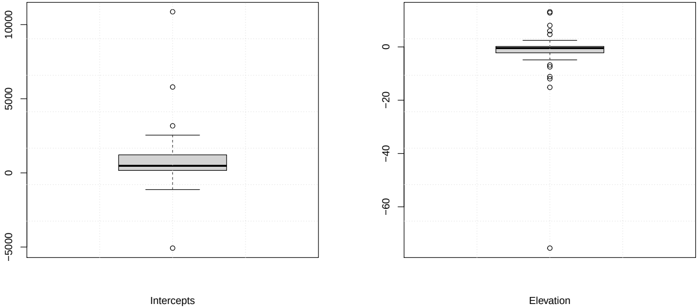
An Example from the EMAP Lake Survey Data

- 334 lakes selected from the population of 21,026 lakes
- 86 Hydrologic Unit Codes (HUCs) are in-sample
- 27 HUCs are out-of-sample
- Estimation of average Acid Neutralising Capacity (ANC) by HUC is of interest.

An Example from the EMAP Lake Survey Data (Cont'd)



An Example from the EMAP Lake Survey Data (Cont'd)



Notation

- m small areas with N_i units;
- y_{ij} and $\mathbf{x}_{ij}$ denote the values of the study variable and a $p \times 1$ vector of known auxiliary variables for the jth unit of the ith small area, respectively, with $i = 1, \ldots, m$, $j = 1, \ldots, N_i$;
- Parameter of interest: $\bar{Y}_i = N_i^{-1} \sum_{j=1}^{N_i} y_{ij},\ i = 1, \ldots, m$.
- n_i is the sample size for area i and it is not large enough to support the use of a direct estimator: $\bar{y}_i = n_i^{-1} \sum_{j \in s_i} y_{ij}$, where s_i denotes the part of the sample from the ith small area.

Nested error regression model (NER)

- Nested error regression model for the finite population:

$$y_{ij} = \beta_0 + \mathbf{x}'_{ij}\boldsymbol{\beta} + \gamma_i + \epsilon_{ij},\ i = 1, \ldots, m;\ j = 1, \ldots, N_i,$$

- β_0 and $\boldsymbol{\beta}$ are fixed intercept and regression coefficients, respectively;
- γ_i is a random effect for area i; ϵ_{ij} is the sampling error for the jth observation in the ith area; γ_i and ϵ_{ij} are all assumed to be independent with $\gamma_i \sim N(0, \sigma_\gamma^2)$ and $\epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$, $i = 1, \ldots, m;\ j = 1, \ldots, N_i$;
- The model parameters σ_γ^2 and σ_ϵ^2 are referred to as the variance components.

An extension of NER model

We propose the following extension of NER model:

$$y_{ij} = \beta_0 + \mathbf{x}'_{ij}\boldsymbol{\beta}_i + \gamma_i + \epsilon_{ij},\ i = 1, \ldots, m;\ j = 1, \ldots, N_i,$$

- β_0 is a common intercept term;
- $\boldsymbol{\beta}_i$ is a $p \times 1$ vector of fixed regression coefficients for area i;
- γ_i and ϵ_{ij} are all independent with $\gamma_i \sim N(0, \sigma_\gamma^2)$ and $\epsilon_{ij} \sim N(0, \sigma_{\epsilon i}^2)$.

The Best Predictor (BP)

The best predictor (BP) of $\bar{Y}_i \approx \theta_i = \beta_0 + \bar{\mathbf{X}}_i' \boldsymbol{\beta}_i + \gamma_i$ is given by

$$\begin{aligned} & \hat{\theta}_i^{BP} \\ = \; & (1 - B_i) \left\{ \bar{y}_i + [\beta_0 + (\bar{\mathbf{X}}_i - \bar{\mathbf{x}}_i)' \boldsymbol{\beta}_i] \right\} + B_i(\beta_0 + \bar{\mathbf{X}}_i' \boldsymbol{\beta}_i) \\ = \; & \hat{\theta}_i(\boldsymbol{\phi}_i), \text{ (say)} \end{aligned}$$

where

- $\bar{\mathbf{X}}_i$: population mean for area i
- $\bar{\mathbf{x}}_i$: sample mean for area i
- $B_i = \frac{\sigma^2_{\epsilon i}/n_i}{\sigma^2_{\epsilon i}/n_i + \sigma^2_\gamma}$;
- $\boldsymbol{\phi}_i = (\beta_0, \boldsymbol{\beta}_i, \sigma^2_\gamma, \sigma^2_{\epsilon i})'$;
- An empirical best predictor (EBP) of θ_i can be written as $\hat{\theta}_i^{EBP} \equiv \hat{\theta}_i(\hat{\boldsymbol{\phi}}_i)$.

Data-driven method for model parameter estimation

- For estimating the model parameters ϕ_i, generalized estimating equations (GEE) with area specific tuning parameters are used to improve prediction accuracy.
- Method allows to borrow strength across areas when estimating each area specific vector of parameters.
- For known area specific tuning parameters, our estimating equation method yields consistent estimators of the model parameters.

Measures of uncertainty of EBP

- Parametric bootstrap
- First-order unbiased when tuning parameters are known.
- We deviate from the standard second-order unbiasedness property of mean squared error (MSE) estimators.
- Method can estimate various uncertainty measures (e.g., MSE, RRMSE, CV, etc.)

EMAP Lake Survey Data Analysis

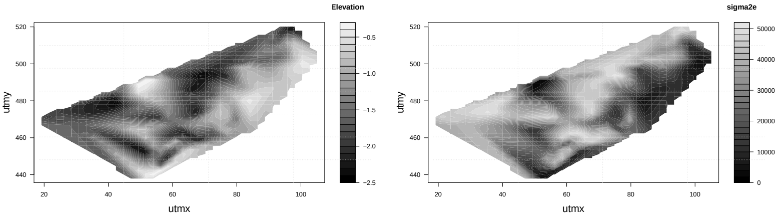
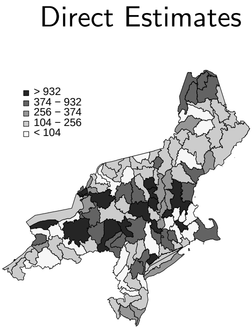
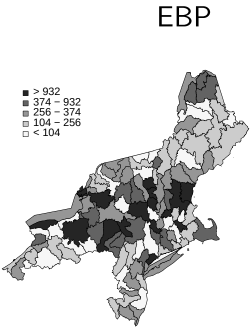


Figure: Maps showing the spatial variation in the HUC-specific area elevation slope coefficient (left) and sampling variance (right) estimates that are generated when the proposed nested error regression model with high dimensional parameter is fitted to the EMAP data.

Maps of estimated average ANC for HUCs using direct and EBP under NERHDP





Boxplot of CVs ratios

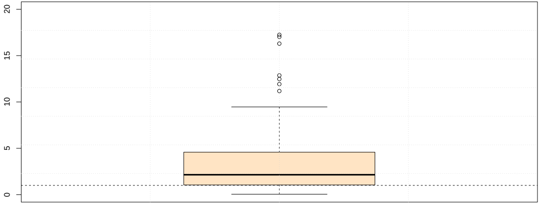


Figure: Boxplot showing the ratio between the CVs of the direct estimates and the CVs of the estimates obtained by the nested error regression model with high dimensional parameter. Values greater than 1 indicates that the CVs of the direct estimates are higher than the other ones.

R package: NERHD

The R package is at:

https://github.com/nicolasalvati73/NERHD/blob/main/NERHD_0.1.1.tar.gz

Figure:

Concluding Remarks

- Flexible modeling
- Area specific estimating equation
- Design consistency
- Starightforward parametric bootstrap for measuring uncertainty
- Method is extendable to estimate nonlinear finite population parameters.

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Thank You!